The Unified Approach to the Construction of Classical Confidence Intervals
How does the famous Feldman-Cousins method work?

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Physicists measure things ($x$)
Measurements have errors
Results reported with errors corresponding to confidence level $x_1(\alpha) \ldots x_2(\alpha)$
(Frequentist) Interpretation

Of an infinite number of identical experiments that measure \( x \) and for which we construct confidence intervals according to a certain prescription, a fraction \( \alpha \) of those intervals would contain the true value (called \( \mu \)).

These intervals say nothing about \( P(\mu|x) \)!!!
These confidence intervals have correct coverage.

- Fraction of intervals containing $\mu$ smaller = undercoverage (serious error!)
- Fraction of intervals containing $\mu$ bigger = overcoverage (too conservative)

If $\mu$ is bound to be positive (e.g. $x$ is a flux), $x_1(\alpha) = 0$ and $x_2(\alpha)$ is reported as the upper limit.
Deciding to report a confidence interval or a measurement after the measurement leads to wrong coverage!

E.g. measurement when at least $3\sigma$ significance is reached, otherwise upper limit

Flip-flopping

E.g. True value 2.5, Gaussian error with $\sigma = 1.0$, report measurement with $1\sigma$ error if $x_0 > 3\sigma$, upper limit for $x_0 < 3\sigma$. Expect true value to be contained in confidence interval in 68.3% of the experiments.

Due to flip-flopping it is only 62%!
Abbildung: Central Confidence interval for Poissonian distribution with $b = 3$.

Abbildung: Upper limits for Poissonian distribution with $b = 3$. 

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Feldman-Cousins Unified approach
Feldman-Cousins Unified Approach

Confidence intervals change smoothly from upper limits to measurements using Neyman’s construction (confidence belts)
Neyman’s Construction provides a range of values for $\mu$ at desired CL when experiment measures an actual $x$.

Calculate confidence interval in $x$ for each possible value of $\mu$.

For each $\mu$ the possible values of $x$ are contained in the confidence interval with a probability $\alpha$.

Measure $x_0$, all $x$-intervals that contain this $x_0$ are part of the confidence interval for $\mu$. 
Abbildung: Neyman Construction of confidence belts.

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Feldman-Cousins Unified approach
How to construct confidence intervals in $x$?

Use Likelihood Ratio to determine what $x$ values should become part of the confidence interval

$$R = \frac{P(x|\mu)}{P(x|\hat{\mu})} = \frac{P(n|s + b)}{P(n|s_{\text{max}} + b)}$$

where $\hat{\mu}$ maximizes $P(x|\mu)$. Calculate $R$ for all $x$ values and rank the $x$ values according to their $R$ values. $x$ values with biggest $R$ are added to the confidence region first until $\int P(x|\mu)dx$ over the confidence region reaches desired confidence level.

$$CL \leq \sum_{n_1, n_2, \ldots, n_c} P(n_i|s + b)$$

where $R_{n_1} > R_{n_2} > R_{n_3} \ldots$
Abbildung: Upper limits and confidence intervals for Poissonian distribution with Feldman-Cousins approach.
Neglecting systematical errors, the limit on the number of signal events $\mu_{CL}$ can be translated into a flux by

$$\Phi_{CL}^{\lim} = \frac{\mu_{CL}}{R_{\nu}(\Phi) t_{obs}} \Phi$$

But a given observation $n$ of events in a search bin is also Poisson distributed with a mean $b$, assuming no signal contribution. The sensitivity is then defined as the average flux upper limit that can be set with 90% confidence.
\[
\left< \mu_{90\%}^{\text{lim}} \right> = \sum_{n=0}^{\infty} \mu_{90\%}^{\text{lim}}(n, b) P(n, b)
\]

This can be translated into an average flux limit

\[
\Phi_{\text{sens}}^{90\%} = \frac{\left< \mu_{90\%}^{\text{lim}} \right>}{R_{\nu}(\Phi) t_{\text{sim}}} \Phi
\]

This is only valid for a specific spectrum!